

Decoy-state quantum key distribution with biased-bases revisited

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In order to improve the key rate of the decoy-state method, we need to jointly study yields of different bases. Given the delicate fact that pulses of the same preparation state can have different counting rates if they are measured in different bases, for example, those vacuum pulses and those single-photon pulses, existing results of decoy-state quantum key distribution using biased bases are actually flawed by assuming that they are equal. We fix this flaw through using the idea that yields of pulses prepared in different bases are same provided that they are prepared in the same state and also they are measured in the same basis, for example, those single-photon pulses prepared in different bases but measured in the same basis. Based on this, we present correct formulas for the decoy-state method using biased bases. Taking the effects of statistical fluctuations into account, we then numerically study the key rates of different protocols with all parameters being fully optimized. Our result confirms the prior art conclusion that decoy-state method using biased bases can have advantage to the symmetric protocol with unbiased bases. We obtain high key rates of our 4-intensity protocol (two in X bases and two in Z bases) without using any vacuum source.

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I. INTRODUCTION

Quantum key distribution (QKD) is one of the most successful applications of quantum information processing. It can help two remote parties, commonly noted as Alice and Bob, to set up the unconditionally secure key. The security of QKD is based on the fundamental laws of quantum physics [1, 2], rather than unproven of computational complexity assumptions.

In practice, imperfect single-photon sources are used in most of the real setups of QKD [1–3]. Such implementations, in principle, suffer from the photon-number-splitting (PNS) attack [4, 5]. The decoy-state method [6–16] and some other methods [17–19] can be used for unconditionally secure QKD even if Alice only uses the imperfect sources [4, 5].

The central issue in the decoy-state method is to faithfully estimate the yield of single-photon pulses. If we do this estimation in each bases separately, i.e., only use those observed outcome of pulses prepared and measured in the same basis, we can obtain two different values of yields, one for the X basis, one for the Z basis. In such a way, results will be faithful but the key rate are not optimized. We can improve the key rate by using the observed results in different bases *jointly*, however, there are delicate points we must pay attention to.

In fact, pulses of the same preparation state can have different counting rates if they are measured in different bases, for example, those vacuum pulses and those single-photon pulses. This point is not considered in all previous biased-basis methods [20–22]. In this paper, we point out that yields of pulses prepared in different bases are same if they are prepared in the same state and measured in the same basis, for example, those single-photon pulses. This is to say, we can treat those measurement outcome from different preparation bases *jointly* provided that they are measured in the same basis. Based on this, we present correct formulas for the decoy-state method using biased bases. We also consider the effects of statistical fluctuations.

In order to make a good estimation of the final key rate with the decoy-state methods, we need to find out the lower and upper bounds of the yield s_0 caused by the vacuum state first. These bounds can be easily obtained if we assume that Alice can prepare a vacuum source. However, in practice, the different intensities are usually generated with an intensity modulator, which has a finite extinction ratio. So it is usually difficult to create a perfect vacuum state in decoy-state QKD experiments [14]. In this paper, we also make a study on the decoy-state method without using vacuum. This is particularly important for the practical implementations.

Regarding the yield s_0 of the vacuum state as a common variable, the final key rate is a function of s_0 . There is a possible region for this values. To extract the final secret key, we should find out the worst-case key rate over the whole region of s_0 values. While in some previous works [21, 22], they straightly use the lower bound of s_0 to calculate the final key rate. Actually, in some cases shown in Sec. III, we find that the smallest value of the final key rate does not appear with the lower bound of s_0 . In this case, if we simple mindedly calculate the final key rate with the lower bound of s_0 , the security of the decoy-state method will not be guaranteed.

The rest of this paper is organized as follows. In Sec. II, we review some delicate points of the decoy-state method. Subsequently, we show the correct formulas to estimate the lower bound of s_1 and the upper bound of e_1 with the effect of statistical fluctuations being taken into account. After that, in Sec. III, we numerically study the optimized key rates of different protocols. The article is ended in Sec. IV with a concluding remark.

II. SOME DELICATE POINTS OF THE DECOY-STATE METHOD AND THE CORRECT FORMULAS

Consider the case we use two bases for state preparation and measurement, the Z basis and the X basis. We denote $s_{1,\alpha}^\omega$ for the yield of those single-photon pulses which are prepared in the α basis and measured in the ω basis. Note that the state of single-photon pulses prepared in the Z basis is same with state of single-photon pulses prepared in the X basis. The density matrix of both states are simply $I/2$. We also denote s_0^ω as the yield of those vacuum pulses which are measured in the ω basis. The delicate point is that even in the asymptotic case

$$s_0^Z \neq s_0^X, \quad (1)$$

and

$$s_{1,\alpha}^Z \neq s_{1,\beta}^X. \quad (2)$$

These are different from the assumptions in prior art works such as [21] where one simply chose to do the decoy-state analysis *only* in Z basis and then just used the value of $s_{1,Z}^Z$ for the value $s_{1,X}^X$. The reason $s_{1,Z}^Z \neq s_{1,X}^X$ is simply due to the mismatch of detection efficiencies and dark counts in different bases. Such mismatch can come from either imperfect control two the devices inside Labs, or Eve's attack [23–26]. If we do the decoy-state study in each basis separately with unbiased bases, we can obtain the lower bound values in each bases separately for $s_{1,Z}^Z$ and $s_{1,X}^X$. The results are faithful but the key rate will be low. Note that here we must use s_0^Z and s_0^X separately in calculating $s_{1,Z}^Z$ and $s_{1,X}^X$.

Here we have a better treatment and we can still study the decoy-state method *jointly* in different bases. For this goal, *we shall use the observed number of counts of pulses prepared in one basis but measured in another basis*. In particular, we have the following elementary equalities

$$s_{1,Z}^Z = s_{1,X}^Z, \quad (3)$$

and

$$s_{1,Z}^X = s_{1,X}^X. \quad (4)$$

Here, $s_{1,\alpha}^\omega(\alpha, \omega = Z, X)$ is the yield of those single-photon pulses which are prepared in the α basis and measured in the ω basis. Eqs. (3, 4) are the main idea of this paper. Given these equations, we *don't* have to study the decoy-state method completely separately in each bases. For example, consider a protocol using a vacuum source O , one source X_1 in the X basis and two sources $\{Z_j | j = 1, 2\}$ in the Z basis. Observing S_O^ω and $\{S_{Z_j}^\omega | j = 1, 2\}$ we can formulate the yield of single-photon pulses measured in the Z basis and also the yield of single-photon pulses measured in the X basis. Here S_O^ω is the yield of the vacuum source measured in the ω basis and $S_{Z_j}^\omega$ is the yield of source Z_j measured in the ω basis. Using the observed values $\{S_{Z_j}^\omega | j = 1, 2\}$ and S_O^ω . Asymptotically we have

$$s_{1,Z}^\omega \geq s_{1,Z}^{\omega,L} = \frac{a_{2,Z_2} S_{Z_1}^\omega - a_{2,Z_1} S_{Z_2}^\omega - A_{Z_1,Z_2}^{0,2} S_O^\omega}{A_{Z_1,Z_2}^{1,2}}, \quad (5)$$

where $A_{Z_1,Z_2}^{0,2} = a_{0,Z_1} a_{2,Z_2} - a_{0,Z_2} a_{2,Z_1}$, $A_{Z_1,Z_2}^{1,2} = a_{1,Z_1} a_{2,Z_2} - a_{1,Z_2} a_{2,Z_1}$ and a_{k,Z_j} are non-negative parameters of sources Z_j with $\rho_{Z_k} = \sum_k a_{k,Z_j} |k\rangle\langle k|$. Here ω can take both X and Z . Given Eqs. (3, 4), the error rate of single-photon pulses prepared and measured in the X basis is

$$e_1^X \leq e_1^{X,U} = \frac{T_{X_1}^X - a_{0,X_1} s_0^X / 2}{a_{1,X_1} s_{1,X}^{X,L}} = \frac{T_{X_1}^X - a_{0,X_1} S_O^X / 2}{a_{1,X_1} s_{1,Z}^{X,L}}, \quad (6)$$

where $T_{X_1}^X$ are error yield of the source X_1 measured in the X basis, a_{k,X_1} are non-negative parameters of source X_1 with $\rho_{X_1} = \sum_k a_{k,X_1} |k\rangle\langle k|$ and the lower bound $s_{1,Z}^{X,L}$ is given by Eq. (5) already. Note that our formulas Eqs. (5,

6) are unconditionally correct under whatever situation, e.g., detection efficiency mismatch, no matter it comes from the imperfect control in side Lab. or Eve.'s attack outside Lab [23–26].

Most generally, Alice can use several different sources. We assume Alice prepares the vacuum source O , 2 non-vacuum sources in the Z basis, Z_1, Z_2 and 2 different non-vacuum sources in the X basis X_1, X_2 with probabilities p_O, p_{Z_1}, p_{Z_2} and p_{X_1}, p_{X_2} , respectively. The density matrices of these non-vacuum sources in photon number space are denoted as follows

$$\rho_{\alpha_l} = \sum_k a_{k,\alpha_l} |k\rangle\langle k|, \quad (7)$$

where a_{k,α_l} are non-negative parameters and α can be the Z and X bases. Meanwhile, we introduce the following very important conditions for two different sources α_1 and α_2

$$\frac{a_{k,\alpha_2}}{a_{k,\alpha_1}} \geq \frac{a_{1,\alpha_2}}{a_{1,\alpha_1}} \geq \frac{a_{0,\alpha_2}}{a_{0,\alpha_1}}, \quad (8)$$

for all $k \geq 2$. We denote $\alpha_1 \prec \alpha_2$ when the sources α_1 and α_2 fulfill the relations presented in Eq.(8). Imperfect sources used in practice such as the coherent state source, the heralded source out of the parametric-down conversion, satisfy the above conditions.

In the protocol, we also assume that Bob measures the received pulses in the Z and X bases with probabilities q^Z and q^X respectively. After the preparation and measurement of N_t pulses, Alice and Bob obtain the observable $N_{\alpha_j}^\omega$ and $M_{\alpha_j}^\omega$ which are the number of successful counts and error counts when Alice sends the pulses from source α_j and Bob measures them in the ω basis (when preparation basis and measured basis are different, we do not need error counts). Here α and ω can take both X and Z . We also denote $S_{\alpha_j}^\omega$ and $T_{\alpha_j}^\omega$ as the yield and error yield respectively with $S_{\alpha_j}^\omega = N_{\alpha_j}^\omega / (p_{\alpha_j} q^\omega N_t)$ and $T_{\alpha_j}^\omega = M_{\alpha_j}^\omega / (p_{\alpha_j} q^\omega N_t)$. As shown, we have the normalized relations $p_O + \sum_{l=1}^2 p_{Z_l} + \sum_{r=1}^2 p_{X_r} = 1$ and $q^X + q^Z = 1$.

Following the GLLP security analysis [3], the final key rate for the source ρ_{Z_l} is given by

$$R = p_{Z_l} q^Z \left\{ a_{1,Z_l} s_1^Z [1 - H(e_1^{p,Z})] - f_e S_{Z_l}^Z H(E_{Z_l}^Z) \right\}, \quad (9)$$

if the source is not used for error test. Here $p_{Z_l} q^Z$ is the raw data sift factor, including the basis-sift factor q^Z and the signal-state ratio p_{Z_l} ; s_1^Z and $e_1^{p,Z}$ are the yield and phase error rate of the single-photon state measured in the Z basis; f_e is the efficiency factor of the error-correction method used; S_{Z_l} and E_{Z_l} are the yield and quantum bit error rate of the source Z_l measured in the Z basis; $H(x) = -x \log_2(x) - (1-x) \log_2(1-x)$ is the binary Shannon entropy function. The phase error rate $e_1^{p,Z}$ can be estimated from the error rate in the X basis while it can not be measured directly.

In any real experiment, the total pulses sent by Alice is finite. In order to extract the secret final key, we have to consider the effect of statistical fluctuations caused by the finite-size. In this case, yields of the same state out of different sources are not always rigorously equal to each other, i.e. $s_{k,\alpha_1}^\omega \neq s_{k,\alpha_2}^\omega$. Here s_{k,α_j}^ω is the yield of k -photon pulses prepared from source α_j and measured in the ω basis. To obtain the lower bound of s_1^ω , one can implement the idea of Ref. [16], i.e. using the averaged yield of a specific state from different sources. As shown, one can introduce the averaged value for the yield of k -photon pulses from all sources that are *prepared in the same state and measured in the same basis*. We define

$$\begin{aligned} \langle s_{k,\alpha}^\omega \rangle &= \frac{1}{c_k^\omega} \sum_l p_{Z_l} a_{k,Z_l} s_{k,Z_l}^\omega \\ &\quad + \frac{1}{c_k^\omega} \sum_r p_{X_r} a_{k,X_r} s_{k,X_r}^\omega, \quad (k = 0, 1); \end{aligned} \quad (10)$$

$$\langle s_{k,\alpha}^\omega \rangle = \frac{1}{c_{k,\alpha}^\omega} \sum_j p_{\alpha_j} a_{k,\alpha_j} s_{k,\alpha_j}^\omega, \quad (k \geq 2), \quad (11)$$

where $c_k^\omega = \sum_l p_{Z_l} a_{k,Z_l} + \sum_r p_{X_r} a_{k,X_r}$ ($k = 0, 1$) and $c_{k,\alpha}^\omega = \sum_j p_{\alpha_j} a_{k,\alpha_j}$ ($k \geq 2$). With Eq. (10), we have $\langle s_{k,X}^\omega \rangle = \langle s_{k,Z}^\omega \rangle$ for $k = 0, 1$. Therefore we shall omit subscript α there for $k = 0, 1$ and use notations $\langle s_0^\omega \rangle$, $\langle s_1^\omega \rangle$ for $\langle s_{0,\alpha}^\omega \rangle$, $\langle s_{1,\alpha}^\omega \rangle$ for simplicity. Also, we define quantity

$$\langle S_{\alpha_l}^\omega \rangle = \sum_{k=0}^{\infty} a_{k,\alpha_l} \langle s_{k,\alpha}^\omega \rangle. \quad (12)$$

Considering the relations in Eq. (12) prepared in the α basis and measured in the ω basis, we can lower bound $\langle s_{1,\alpha}^\omega \rangle$ for a given value of $\langle s_0^\omega \rangle$ with the following equations [15]

$$\langle s_{1,\alpha}^\omega \rangle \geq \langle s_1^{\omega,L} \rangle = \max_{\alpha=Z,X} \left[\langle s_{1,\alpha}^{\omega,L} \rangle \langle s_0^\omega \rangle \right] \quad (13)$$

and

$$\langle s_{1,\alpha}^{\omega,L} \rangle \langle s_0^\omega \rangle = \frac{a_{2,\alpha_2} \underline{S}_{\alpha_1}^\omega - a_{2,\alpha_1} \overline{S}_{\alpha_2}^\omega - A_{\alpha_1\alpha_2}^{0,2} \langle s_0^\omega \rangle}{A_{\alpha_1\alpha_2}^{1,2}}, \quad (14)$$

where $A_{\alpha_1\alpha_2}^{0,2} = a_{0,\alpha_1} a_{2,\alpha_2} - a_{0,\alpha_2} a_{2,\alpha_1}$, $A_{\alpha_1\alpha_2}^{1,2} = a_{1,\alpha_1} a_{2,\alpha_2} - a_{1,\alpha_2} a_{2,\alpha_1}$ and

$$\underline{S}_{\alpha_j}^\omega = S_{\alpha_j}^\omega / (1 + \delta_{\alpha_j}^\omega), \quad \overline{S}_{\alpha_j}^\omega = S_{\alpha_j}^\omega / (1 - \delta_{\alpha_j}^\omega). \quad (15)$$

By using the multiplicative form of Chernoff bound [27, 28], with a fixed failure probability ϵ , we can give an interval of $\langle S_{\alpha_j}^\omega \rangle$ with the observable $S_{\alpha_j}^\omega$, $[\underline{S}_{\alpha_j}^\omega, \overline{S}_{\alpha_j}^\omega]$, which can bound the value of $\langle S_{\alpha_j}^\omega \rangle$ with a probability at least $1 - \epsilon$. Similarly, we can also define $\overline{T}_{\omega_j}^\omega$ with the observable $T_{\omega_j}^\omega$. Note that $\langle s_1^{\omega,L} \rangle$ is actually a function of $\langle s_0^\omega \rangle$.

With the mean values $\langle s_1^{\omega,L} \rangle$ defined in Eq. (13), the lower bounds of s_{1,Z_l}^Z and s_{1,X_r}^X can be estimated by

$$s_{1,Z_l}^{Z,L} = \langle s_1^{Z,L} \rangle (1 - \delta_{Z_l}), \quad s_{1,X_r}^{X,L} = \langle s_1^{X,L} \rangle (1 - \delta_{X_r}), \quad (16)$$

where $\delta_{Z_l} = \lambda / \sqrt{N_{1,Z_l}^Z \langle s_{1,Z_l}^{Z,L} \rangle}$ with $\lambda = \sqrt{-2 \ln \epsilon}$ and $\delta_{X_r} = \lambda / \sqrt{N_{1,X_r}^X \langle s_{1,X_r}^{X,L} \rangle}$. Here and after, we define

$$N_{k,\alpha_j}^\omega = a_{k,\alpha_j} p_{\alpha_j} q^\omega N_t, \quad (17)$$

as the number of k -photon pulses prepared in source α_j and measured in the basis ω .

In order to estimate the final key rate, we also need the upper bound of the error rate e_1^X . Similarly, we have

$$e_{1,X_1}^X \leq e_{1,X_1}^{X,U} = \frac{T_{X_1}^X - a_{0,X_1} \langle s_0^X \rangle (1 - \delta_{0,X_1}^X) / 2}{a_{1,X_1} s_{1,X_1}^{X,L}}, \quad (18)$$

where $\delta_{0,X_1}^X = \lambda / \sqrt{N_{0,X_1}^X \langle s_0^X \rangle}$.

If the key size is infinite, the phase-flip error rate for single-photon counts in Z basis is simply $e_1^{p,Z} = e_1^X$. In a finite-key-size case, we can apply the large data size approximation of the random sampling method [30] to upper bound the phase error rate $e_1^{p,Z}$ of single-photon pulses prepared and measured in the Z basis with the failure probability ϵ

$$e_1^{p,Z} \leq e_1^{p,Z,U} = e_{1,X_1}^{X,U} + \theta_Z^X, \quad (19)$$

where $\theta_Z^X = \sqrt{n_\theta / d_\theta}$ with $d_\theta = \frac{(1-g_X)g_X \ln 2}{2(1-e_1)e_1}$, $n_\theta = -\log[\epsilon \sqrt{e_1(1-e_1)n_X n_Z / (n_X + n_Z)}] / (n_X + n_Z)$ and $g_X = \frac{n_X}{n_X + n_Z}$. Here we write $n_X = N_{1,X_1}^X$, $n_Z = \sum_l N_{1,Z_l}^Z$ and $e_1 = e_{1,X_1}^{X,U}$ for simplicity. Note that $e_1^{p,Z,U}$ is a function of $\langle s_0^X \rangle$. Straightly, we can also formulate the upper bound of the phase-flip error rate of single-photon counts in X basis, being denoted by $e_1^{p,X,U}$. We omit the explicit formula here since it is just trivially written analogically to Eq.(19).

Note that $\langle s_0^X \rangle$ (or $\langle s_0^Z \rangle$) is the common variable in both quantity $\langle s_1^{X,L} \rangle$ and $e_1^{p,Z,U}$ (or quantity $\langle s_1^{Z,L} \rangle$ and $e_1^{p,X,U}$) shown in Eq.(13) and Eq.(19) respectively. We need to know the range of this for final key calculation.

If Alice uses a vacuum source in the protocol, the bounds are simply

$$\overline{S}_O^\omega = \langle s_0^{\omega,U} \rangle \geq \langle s_0^\omega \rangle \geq \langle s_0^{\omega,L} \rangle = \underline{S}_O^\omega. \quad (20)$$

However, in practice, it is usually difficult to create a perfect vacuum state in decoy-state QKD experiments [14]. We may consider the decoy-state methods without using a vacuum source. In the method, Eq.(20) can not be used directly.

Reconsidering the relations in Eq. (12) with $j = 1, 2$, we can lower bound $\langle s_0^\omega \rangle$ by eliminating $\langle s_{1,\alpha}^\omega \rangle$

$$\langle s_0^\omega \rangle \geq \langle s_0^{\omega,L} \rangle(\alpha) = \frac{a_{1,\alpha_2} \underline{S}_{\alpha_1}^\omega - a_{1,\alpha_1} \overline{S}_{\alpha_2}^\omega}{A_{\alpha_1\alpha_2}^{0,1}}, \quad (21)$$

where $A_{\alpha_1\alpha_2}^{0,1} = a_{0,\alpha_1}a_{1,\alpha_2} - a_{0,\alpha_2}a_{1,\alpha_1}$. With these preparations, in the method without the assumption of vacuum, we can write the lower bound of $\langle s_0^\omega \rangle$ as

$$\langle s_0^\omega \rangle \geq \langle s_0^{\omega,L} \rangle = \max_{\alpha=X,Z} \left\{ \langle s_0^{\omega,L}(\alpha) \rangle, 0 \right\}. \quad (22)$$

By simply attributing all the errors to the vacuum pulses, we can upper bound $\langle s_0^\omega \rangle$ with

$$\langle s_0^\omega \rangle \leq \langle s_0^{\omega,U} \rangle = \min \left\{ 2\bar{T}_{\omega_1}^\omega / a_{0,\omega_1}, \bar{S}_{Z_1}^\omega / a_{0,Z_1}, \bar{S}_{X_1}^\omega / a_{0,X_1} \right\}. \quad (23)$$

III. NUMERICAL SIMULATION

A. Some special protocols

Choosing different number of sources in the X and Z bases, we list some protocols in Tab. I.

Protocol	Description
3Int-0	$O, Z_1, Z_2, X_1, X_2; Z_j = X_j, p_{Z_j} = p_{X_j}, q^Z = q^X.$
3Int-1	$O, Z_1, Z_2, X_1; Z_1 = X_1, Z_1 \prec Z_2, \mu_{Z_2} = 0.479.$
4Int-1	$O, Z_1, Z_2, X_1; Z_1 \prec Z_2.$
4Int-2	$Z_1, Z_2, X_1, X_2; Z_1 \prec Z_2, X_1 \prec X_2.$
5Int-1	$O, Z_1, Z_2, X_1, X_2; Z_1 \prec Z_2, X_1 \prec X_2.$

TABLE I: List of some practical decoy-state methods. O is the vacuum source. Z_l and X_r are the sources used by Alice in the Z and X bases respectively. The sources are different from each other except when we write the equivalent relations in the table, such as $Z_j = X_j$. The probabilities of choosing the sources are independent except for the normalized condition. In 3Int-0, we also assume $Z_1 \prec Z_2$.

The first three-intensity protocol 3Int-0 listed in the table is the original symmetric method discussed in Ref. [14]. In 3Int-0, Alice uses the vacuum source and two different non-vacuum sources in the Z and X bases to prepare the pulses. The symmetric conditions can be described by $Z_j = X_j$, $p_{Z_j} = p_{X_j}$ for $j = 1, 2$ and $q^Z = q^X$ with the notations presented in this paper. In order to estimate the lower bound of the yield of single-photon pulses, we also need to assume $Z_1 \prec Z_2$ which indicates that the sources Z_1 and Z_2 fulfill the relations presented in Eq.(8) with $\sigma_1 = Z_1$ and $\sigma_2 = Z_2$.

The second three-intensity protocol 3Int-1 listed in Tab. I is the method considered in Ref. [21]. This is then further studied in Ref. [22] with one more free intensity while the security loopholes as discussed earlier in this paper is still there and the key rate is not really fully optimized. In 3Int-1, Alice uses the vacuum source, two different sources Z_1, Z_2 in the Z basis and only one source X_1 in the X basis. The intensities of the source in the X basis is equal to the first one in the Z basis, i.e. $Z_1 = X_1$. Furthermore, the intensity (weak coherent state used in the simulation) of the second source in the Z basis is 0.479.

In Tab. I, we also list two different four-intensity protocols. The second one, 4Int-2, is a new general protocol without the assumption of vacuum, whereas the vacuum source are used in the protocol 4Int-1. Taking the protocol 4Int-2 as an example, Alice uses two different sources in the Z and X bases respectively. For these four different sources, we assume $Z_1 \prec Z_2$ and $X_1 \prec X_2$. In this protocol, we use sources Z_2 and X_2 to extract the final key. With Eq.(16), the lower bounds of s_{1,Z_l}^Z and s_{1,X_r}^X can be estimated. The upper bound of $e_1^{p,Z}$ can be calculated with Eq.(19). The upper bound of $e_1^{p,X}$ can also be estimated in the same way. In 4Int-1, the sources Z_2 and Z_1 are used to extract the final key.

Besides these three-intensity and four-intensity protocols, we list a five-intensity protocol 5Int-1 in Tab. I. In 5Int-1, Alice uses the vacuum source, two different sources in the Z and X bases to prepare the pulses. In order to make 5Int-1 contains 4Int-1 and 4Int-2 as its special cases, we need to introduce a probability $p_{Z_1}^s$ with $0 \leq p_{Z_1}^s \leq p_{Z_1}$. With probability $p_{Z_1}^s$, Alice will randomly choose pules from Z_1 to extract the final key.

To maximize the key rates, in 3Int-1 and 4Int-1, we shall use the following economic worst-case estimation formula

$$R = \min_{\langle s_0^X \rangle} [R(\langle s_0^X \rangle)] \quad (24)$$

over the region for all possible values of $\langle s_0^X \rangle$ in $[\langle s_0^{X,L} \rangle, \langle s_0^{X,U} \rangle]$. Here function R is $R = R_1 + R_2$ and

$$R_l = p_{Z_l} q^Z \left\{ a_{1,Z_l} s_{1,Z_l}^{Z,L} [1 - H(e_1^{p,Z,U})] - f_e S_{Z_l}^Z H(E_{Z_l}^Z) \right\}. \quad (25)$$

While in the 3Int-0, 4Int-2 and 5Int-1, we use the following economic more worst-case estimation for key rates

$$R = \min_{\langle s_0^Z \rangle, \langle s_0^X \rangle} [R(\langle s_0^Z \rangle, \langle s_0^X \rangle)] \quad (26)$$

over the region for all possible values of $\langle s_0^Z \rangle$ and $\langle s_0^X \rangle$ in $[\langle s_0^{Z,L} \rangle, \langle s_0^{Z,U} \rangle]$ and $[\langle s_0^{X,L} \rangle, \langle s_0^{X,U} \rangle]$ respectively. Here

$$R(\langle s_0^X \rangle, \langle s_0^Z \rangle) = R_X + R_Z \quad (27)$$

and

$$R_Z = p_{Z_2} q^Z \left\{ a_{1,Z_2} s_{1,Z_2}^{Z,L} [1 - H(e_1^{p,Z,U})] - f_e S_{Z_2}^Z H(E_{Z_2}^Z) \right\}, \quad (28)$$

$$R_X = p_{X_2} q^X \left\{ a_{1,X_2} s_{1,X_2}^{X,L} [1 - H(e_1^{p,X,U})] - f_e S_{X_2}^X H(E_{X_2}^X) \right\}. \quad (29)$$

Note that in such a case, we need to calculate the final key rate with two variables, $\langle s_0^Z \rangle$ and $\langle s_0^X \rangle$ *jointly*. If not using this trick, the simple worst-case treatment that treats each one separately will produce a lower key rate.

In subsection III B, we will compare the results for these protocols with the full parameters optimization. After that, we will see that the final key rates obtained with 4Int-2 and 5Int-1 are nearly equal to each other. That is to say, it is advantageous to use the general protocol 4Int-2 in practice, as it can give an almost optimal key rate and has no use for the vacuum source.

B. Numerical results

In this subsection, we will present some results of the numerical simulation. We also optimize all parameters by the method of full optimization. For a fair comparison, we use all the sifted key corresponding to the successful events obtained with X and Z bases to extract the final key. We shall estimate what values would be observed for the yields and error yields in the normal cases by the linear channel loss model shown in appendix A. We use the same experimental parameters used in Ref.[29] for our numerical simulation, which are also used for simulation in Ref.[21]. The values of these parameters are listed in Table II. In the simulation, we also assume that Alice uses the coherent states to prepare the pulses. Then the yields $S_{\alpha_j}^\omega$ and error yields $T_{\alpha_j}^\omega$ can be calculated with different intensities. By using these values, we can estimate the lower bound of $s_{1,Z_l}^Z(s_{1,X_r}^X)$ and the upper bound of $e_1^{p,Z}(e_1^{p,X})$ with the method presented in Sec. II.

e_d	p_d	η_d	f_e	ϵ	α
3.3%	1.7×10^{-6}	4.5%	1.16	10^{-10}	0.2

TABLE II: List of experimental parameters used in numerical simulations. e_d : the misalignment-error probability, p_d : the background counting rate, η_d : the detector efficiency, f_e : the error correction inefficiency, ϵ : the security bound considered in the finite-data analysis, i.e., failure probability, α : the loss coefficient of the standard fiber measured in dB/km.

To make a fair comparison, we make the full parameter optimization for all protocols. Here we also use the well-known local search algorithm [31]. In this algorithm, we need to optimize the one-variable nonlinear function in each step for the local search. In the optimization, except for those bits for error test, all bits are used for final key distillation. In particular, for protocols 3Int-1, 4Int-1, all bits due to sources Z_1, Z_2 are used for final key distillation; while in protocols 3Int-0, 4Int-2, bits due to sources Z_2, X_2 are used for final key distillation. In order to make the protocol 5Int-1 contains 4Int-1 and 4Int-2 as its special cases, besides bits due to sources Z_2, X_2 being used for final key distillation, we also need to split the bits due to source Z_1 into two parts (one for error test and the other for final key distillation) by introducing a probability $p_{Z_1}^s$ with $0 \leq p_{Z_1}^s \leq p_{Z_1}$.

We consider the different methods in the case of data-size $N_t = 10^{10}$. The optimal key rates of per pulse for the distances 80km, 90km, 100km 110km and 120km (standard fiber), with the statistical fluctuations, are shown in Table III. Comparing the results with the original symmetric protocol 3Int-0 and the biased-basis protocol 3Int-1 discussed in Ref. [21], the achievable key rate can be significantly improved with our new protocols.

	80km	90km	100km	110km	120km
3Int-0	3.37e-5	1.73e-5	7.65e-6	2.39e-6	3.91e-8
3Int-1	5.04e-5	2.38e-5	9.29e-6	2.02e-6	0
4Int-1	5.43e-5	2.65e-5	1.09e-5	2.99e-6	3.65e-8
4Int-2	5.05e-5	2.39e-5	9.63e-6	3.50e-6	4.55e-7
5Int-1	5.43e-5	2.65e-5	1.09e-5	3.50e-6	4.55e-7

TABLE III: Comparison of the optimal final key rates at different distances (standard fiber) with statistical fluctuation analysis in the case of data-size $N_t = 10^{10}$ (total number of pulses). Results for methods in Tab. I are listed here. Comparing with the results obtained with the original symmetric protocol 3Int-0 and the biased-basis protocol 3Int-1 presented in Ref. [21], our protocols can significantly improve the the key rates. The key rate of our 4Int-2 protocol which does not use vacuum is very close to that of the 5Int protocol. Interestingly, the 5Int protocol automatically comes to one of the 4-intensity protocol at each distances. This strongly indicates that we have indeed reached the key rate maximization. This also shows that, for a given distance, instead of using the 5 intensity protocol, just choosing one of the 4 intensities is enough.

In Tab. IV, we list the value of parameters with all of them being optimized. In the 1st column, we show the parameters for the original symmetric protocol 3Int-0. In the 2nd column, the parameters for the 4-intensity protocol 4Int-1 are listed. In the last column, we exhibit the parameters for our new 4-intensity protocol 4Int-2.

3Int-0	4Int-1	4Int-2
$p_{Z_2} = p_{X_2} = 0.338$	$p_{Z_2} = 0.597$	$p_{Z_2} = 0.260$
$p_{Z_1} = p_{X_1} = 0.142$	$p_{Z_1} = 0.190$	$p_{Z_1} = 0.077$
$p_O = 0.040$	$p_{X_1} = 0.112$	$p_{X_2} = 0.458$
	$p_O = 0.101$	$p_{X_1} = 0.205$
$\mu_{Z_2} = \mu_{X_2} = 0.390$	$\mu_{Z_2} = 0.379$	$\mu_{Z_2} = 0.419$
$\mu_{Z_1} = \mu_{X_1} = 0.116$	$\mu_{Z_1} = 0.078$	$\mu_{Z_1} = 0.200$
$\mu_O = 0$	$\mu_{X_1} = 0.255$	$\mu_{X_2} = 0.396$
	$\mu_O = 0$	$\mu_{X_1} = 0.073$
$q^X = 0.5$	$q^X = 0.223$	$q^X = 0.579$

TABLE IV: Comparison of parameters for the distance 110km (standard fiber) with statistical fluctuation analysis in the case of data-size $N_t = 10^{10}$ for different protocols. In the 1st, 2nd and 3rd columns we list the value of parameters with all of them being optimized for 3Int-0, 4Int-1 and 4Int-2 respectively. p_{α_j} is the probability to use the source α_j in the protocol. p_O is the probability to choose the vacuum source. μ_{α_j} is the intensity of the coherent source α_j . q^X is the probability that Bob measures the received pulses in the X basis.

More extensive results are shown in Fig. 1. In this figure, we show the optimal key rate (per pulse) in logarithmic scale as a function of the distance under a practical setting with finite data-set $N_t = 10^{10}$. In Fig. 1, we use the thin magenta dash-dot line, the thin black solid line, the blue dashed line, the red solid line and the green dash-dot line to indicate the results obtained with the protocols 3Int-0, 3Int-1, 4Int-1, 4Int-2 and 5Int-1 respectively. From the numerical simulation, we can conclude that both the achievable key rate and distance can be significantly improved by using our new protocols compared with 3Int-0 and 3Int-1. The optimal final key rates obtained by using the four-intensity protocol 4Int-2 and the five-intensity protocol 5Int-1 are nearly equal to each other. In practical QKD applications, for better performance in terms of key rate and distance, it is advantageous to make use of 4Int-2 with full parameter optimization, because of the advantage of not using any vacuum source.

As discussed in Sec. II, in order to estimate the final secure key rate, we need to find out the smallest one with variable $\langle s_0^X \rangle$ changing in the region $[\langle s_0^{X,L} \rangle, \langle s_0^{X,U} \rangle]$. Previously, in Refs. [20, 21], they straightly use the lower bound $\langle s_0^{X,L} \rangle$ to calculate the final key rate. In order to insure the security, we must make sure that the smallest key rate value just happened at the lower bound value of $\langle s_0^X \rangle$ when we use it to calculate the final key rate directly. However, in the numerical simulation, we find some counterexamples. In Fig. 2, after evaluating $\langle s_0^Z \rangle$ with its proper value $\langle \tilde{s}_0^Z \rangle$, we plot the key rate R as a function of $\langle s_0^X \rangle$ with $\langle s_0^X \rangle \in [\langle s_0^{X,L} \rangle, \langle s_0^{X,U} \rangle]$ for the protocol 4Int-2 at the distance

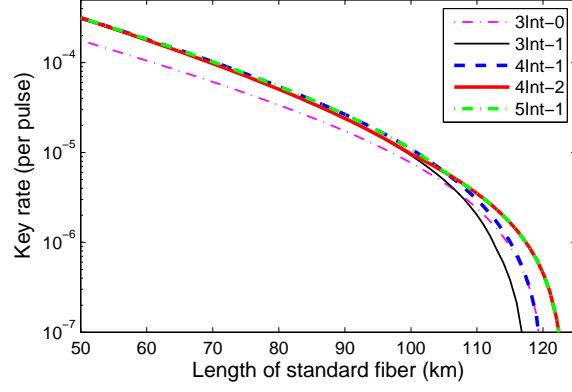


FIG. 1: (Color online) Optimal secure key rates (per pulse) in logarithmic scale as a function of the distance under different protocols. Here we set $N_t = 10^{10}$. The key rates are obtained by using numerical methods with 3Int-0 (thin magenta dash-dot line), 3Int-1 (thin black solid line), 4Int-1 (blue dashed line), 4Int-2 (red solid line) and 5Int-1 (green dash-dot line). The key rates with four-intensity protocol 4Int-2 and five-intensity protocol 5Int-1 are almost overlapped. In simulation, we perform a full parameter optimization for all cases.

of 100km. From this figure, we can see that the smallest value of $R(\langle s_0^X \rangle) = R(\langle s_0^X \rangle, \langle \tilde{s}_0^Z \rangle)$ does not obtained with $\langle s_0^X \rangle = \langle s_0^{X,L} \rangle$. In this case, we can not calculate the final key rate with $\langle s_0^{X,L} \rangle$ directly. That is to say, if we use the lower bound of the yield of vacuum state to calculate the final key rate in this counterexample, the security will not be guaranteed.

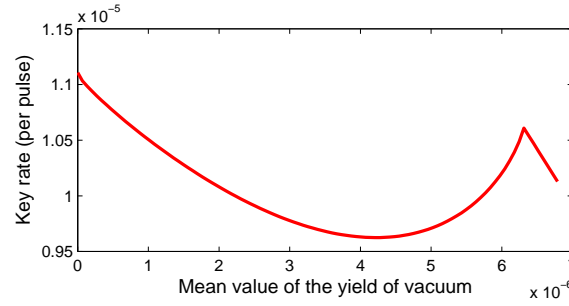


FIG. 2: (Color online) The key rate $R(\langle s_0^X \rangle) = R(\langle s_0^X \rangle, \langle \tilde{s}_0^Z \rangle)$ vs the yield of vacuum state as the variable and $\langle s_0^X \rangle \in [\langle s_0^{X,L} \rangle, \langle s_0^{X,U} \rangle]$ with 4Int-2 for the distance 100km (standard fiber). From this figure, we can see that the smallest values of R is not obtained at the lower bounds of $\langle s_0^X \rangle$.

IV. CONCLUSION

We propose to make use of the yields of those pulse which are prepared in one basis and measured in another basis. This is to say, we can treat those measurement outcome from different preparation bases *jointly* provided that they are measured in the same basis. Based on this, we present correct formulas for the decoy-state method with considering the effects of statistical fluctuations.

In practice, it is usually difficult to create a perfect vacuum state in decoy-state QKD experiments. In this paper, we also present an analytical approach with general decoy states, i.e. without the assumption of vacuum. By taking the full parameter optimization with the linear channel loss model, we find that our new 4-intensity protocol 4Int-2 can give an almost optimal key rate. The vacuum source and more decoy states cannot help to increase the key rate.

In the simulation, we find some counterexamples in which the lower bound of the final key rate does not obtained with the lower bound of the yield of the vacuum state. That is to say, if we use the lower bound of the yield of the vacuum state to calculate the final key rate directly in the counterexample, the security will not be guaranteed. So we need to find out the smallest key rate with different values of the yield of the vacuum state faithfully.

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Appendix A: Linear Channel Loss Model

In order to do the numerical simulations, we need the observed yields and error yields. Here we consider a widely used linear channel loss model [8]. With this model, the yields and error yields can be simulated. In this model, Alice randomly chooses the states for different sources with corresponding probabilities.

For this model, we define η^ω as the overall transmission of the channel when Bob measures the pulses in the ω basis ($\omega = X, Z$). It should be note that η^ω includes the detector efficiency and is measured in dB. In the real setups, such as in the situation with different detector efficiencies, the transmissions η^X and η^Z are not equal to each other rigorously. So we should treat the yields separately with different bases used by Bob to measure the pulses.

It is reasonable to assume independent between the behaviors of the k photons in a k -photon state. Therefore the transmittance of the k -photon state η_k^ω is given by

$$\eta_k^\omega = 1 - (1 - \eta^\omega)^k, \quad (k = 0, 1, 2, \dots) \quad (\text{A1})$$

when Bob measures the pulses in the ω basis.

Define $s_{k,\alpha}^\omega$ to be the yield of an k -photon state when Alice prepares the state in the α basis and Bob measures in the ω basis ($\omega, \alpha = X, Z$). Specially, $s_{0,\alpha}^\omega$ is the background rate. It should be note that $s_{0,\alpha}^\omega$ is independent of the α basis used by Alice to prepare the pulses. Then we can write $s_{0,\alpha}^\omega$ into s_0^ω concisely.

As discussed above, we need to treat $s_{k,\alpha}^\omega$ with $\omega = X$ and $\omega = Z$ separately. In the protocol, there are two detectors are used when Bob chooses the ω basis to measure the pulses. Furthermore, Bob denote the situation when one and only detector makes a count as a successful event. The yield of an k -photon state, $s_{k,\alpha}^\omega$, mainly comes from two parts, background and true signal. Assuming that the background counts are independent of the signal photon detection, then $s_{k,\alpha}^\omega$ is given by

$$s_{k,\alpha}^\omega = (1 - s_0^\omega)[1 - (1 - 2s_0^\omega)(1 - \eta^\omega)^k], \quad (\text{A2})$$

when Alice and Bob choose the same bases, i.e. $\omega = \alpha$. Furthermore, we have

$$s_{k,\alpha}^\omega = 2(1 - s_0^\omega)[(1 - \eta^\omega/2)^k - (1 - s_0^\omega)(1 - \eta^\omega)^k], \quad (\text{A3})$$

when the bases used by Alice and Bob are different, i.e. $\omega \neq \alpha$.

In the ideal case without misalignment error, the error rate of the k -photon state when Alice and Bob use the same bases, $\hat{e}_{k,\omega}^\omega$, is given by

$$\hat{e}_{k,\omega}^\omega = s_0^\omega(1 - s_0^\omega)(1 - \eta^\omega)^k. \quad (\text{A4})$$

In the real setups, we denote e_d as the misalignment error probability. Then the error rate of the k -photon state is

$$e_{k,\omega}^\omega = e_d(1 - 2\hat{e}_{k,\omega}^\omega) + \hat{e}_{k,\omega}^\omega. \quad (\text{A5})$$

With these notations, we can write the yields of the pulses prepared from source α_j and measured in the ω basis as

$$S_{\alpha_j}^\omega = \sum_k a_{k,\alpha_j} s_{k,\alpha}^\omega, \quad (\text{A6})$$

where the density matrix of source α_j given by Eq.(7) has been used. The error yield is given by

$$T_{\alpha_j}^\omega = E_{\alpha_j}^\omega S_{\alpha_j}^\omega = [e_d(1 - 2\hat{E}_{\alpha_j}^\omega) + \hat{E}_{\alpha_j}^\omega] S_{\alpha_j}^\omega, \quad (\text{A7})$$

when $\omega = \alpha$ with $\hat{E}_{\alpha_j}^\omega = \sum_k a_{k,\omega_j} e_{k,\omega}^\omega$. Furthermore, the error rate of the k -photon state is always equal to 1/2 when Alice and Bob use the different bases.

If we consider the weak coherent sources, assuming that the phase of each pulse is totally randomized, the density matrix of the coherent state with intensity μ can be written as

$$\rho = \sum_k \frac{e^{-\mu} \mu^k}{k!} |k\rangle \langle k|.$$

With this formula, the yield of the pulses prepared from the weak coherent source α_j with density μ_{α_j} and measured in the ω basis is

$$S_{\alpha_j}^\omega = (1 - s_0^\omega)[1 - (1 - 2s_0^\omega)e^{-\mu_{\alpha_j}\eta^\omega}], \quad (\text{A8})$$

when $\omega = \alpha$ and

$$S_{\alpha_j}^\omega = 2(1 - s_0^\omega)e^{-\frac{\mu_{\alpha_j}\eta^\omega}{2}}[1 - (1 - s_0^\omega)e^{-\frac{\mu_{\alpha_j}\eta^\omega}{2}}], \quad (\text{A9})$$

when $\omega \neq \alpha$. The error yield can be simulated by Eq.(A7) when $\alpha = \omega$ with $\hat{E}_{\omega_j}^\omega = s_0^\omega(1 - s_0^\omega)e^{-\mu_{\omega_j}\eta^\omega}$.

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